Maxmin Strategies

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Maxmin Strategies

- Player i's maxmin strategy is a strategy that maximizes i's worst-case payoff, in the situation where all the other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to i.
- The maxmin value (or safety level) of the game for player i is that minimum payoff guaranteed by a maxmin strategy.

Definition (Maxmin)

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$, and the maxmin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$.

- Why would i want to play a maxmin strategy?

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- Why would i want to play a maxmin strategy?
  - a conservative agent maximizing worst-case payoff
  - a paranoid agent who believes everyone is out to get him

Minmax Strategies

- Player i's minmax strategy against player $-i$ in a 2-player game is a strategy that minimizes $-i$'s best-case payoff, and the minmax value for i against $-i$ is payoff.

Definition (Minmax, 2-player)

In a two-player game, the minmax strategy for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

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- Why would i want to play a minmax strategy?
  - to punish the other agent as much as possible

Minmax Theorem

Theorem (Minimax theorem (von Neumann, 1928))

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

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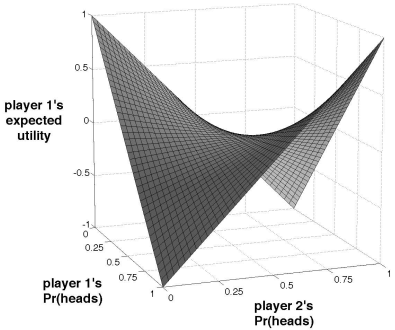
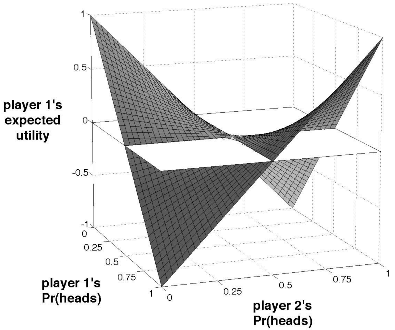
1. Each player's maxmin value is equal to his minmax value. The maxmin value for player 1 is called the value of the game.
2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.
3. Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).

Saddle Point: Matching Pennies





2×2 Zero-sum Games

- Minmax or maxmin produces the same result as method for finding NE in general 2×2 games;
- Check against penalty kick game.

Penalty Kick Game

Goalie

Kicker		L	R
	L	$0.6, 0.4$	$0.8, 0.2$
	R	$0.9, 0.1$	$0.7, 0.3$

How does the kicker maximize his minimum?

Penalty Kick Game

Kicker \ Goalie	L	R
L	$0.6, 0.4$	$0.8, 0.2$
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How does the kicker maximize his minimum?

$$\max_{s_1} \min_{s_2} \left[s_1(L)s_2(L) \cdot 0.6 + s_1(L)s_2(R) \cdot 0.8 + s_1(R)s_2(L) \cdot 0.9 + s_1(R)s_2(R) \cdot 0.7 \right]$$

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$$= \min_{s_2} \left[\begin{array}{l} s_1(L)s_2(L) \cdot 0.6 + s_1(L)(1 - s_2(L)) \cdot 0.8 + \\ + (1 - s_1(L))s_2(L) \cdot 0.9 + (1 - s_1(L))(1 - s_2(L)) \cdot 0.7 \end{array} \right]$$

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		Goalie L	Goalie R
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$$= \min_{s_2} \left[(0.2 - s_1(L)\cdot 0.4)\cdot s_2(L) + (0.7 + s_1(L)\cdot 0.1) \right]$$

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$$= \min_{s_2} [(0.2 - s_1(L) \cdot 0.4) \cdot s_2(L) + (0.7 + s_1(L) \cdot 0.1)]$$

$$\Rightarrow 0.2 - s_1(L) \cdot 0.4 = 0$$

$$\Rightarrow s_1(L) = 1/2, \; s_1(R) = 1/2$$

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How does the goalie minimize the kicker's maximum?

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$$\Rightarrow 0.1 - s_2(L) \cdot 0.4 = 0$$

$$\Rightarrow s_2(L) = 1/4, \;\; s_2(R) = 3/4$$

Computing Minmax

For 2 players minmax is solvable with LP (Linear Programming).

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$$
\begin{aligned}
\text{minimize} \quad & U_1^* \\
\text{subject to} \quad & \sum_{k \in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k \leq U_1^* & \forall j \in A_1 \\
& \sum_{k \in A_2} s_2^k = 1 \\
& s_2^k \geq 0 & \forall k \in A_2
\end{aligned}
$$